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TWO-FLUID ANALYSIS OF CENTRIFUGAL SEPARATION IN A FINITE CYLINDER

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Abstract—The flow of a separating two-phase suspension in a rotating finite axisymmetric cylinder is analyzed. The Ekman suction is reproduced by an integral formula expected to hold when both the Ekman and particle Taylor numbers, E and β, are small. The interior two-fluid averaged equations of motion are subsequently reduced to an initial-value system, whose (numerical) solution points out the effect of the Ekman layers on the separation process. It is confirmed that the global significance of these shear regions is represented by the parameter $\lambda = E^{1/2}/|\epsilon|\beta H$, where H is the dimensionless height of the container and ϵ is the relative density difference between the phases. As compared to the "infinitely long" cylinder ($\lambda = 0$) case, the most essential effects which show up when λ is of order 1 or larger are the following: (a) an O(1) diminution in the angular mixture velocity, measured in the rotating system of the cylinder; (b) an $O(|\epsilon|\alpha(0))$ enhancement of the relative velocity between the phases, where $\alpha(0)$ is the initial volume fraction of the dispersed phase. The present results are in good agreement with those of an earlier mixture (diffusion) formulation when $|\epsilon| \ll 1$, and recover the "infinitely long" cylinder theory when $\lambda \to 0$.

1. INTRODUCTION

The centrifugal separation of a mixture is a complicated process, the understanding of which is of intrinsic theoretical and practical importance. Several pertinent aspects of this subject are reflected in the typical configuration to be considered in the present study, figure 1: the motion of a rotating fluid mixture in a *finite*, straight axisymmetric cylinder of radius r_0^* and length H^* (the upper asterisk denotes dimensional variables). The mixture consists of a dispersed phase of particles (or droplets) of density ρ_D^* and uniform radius a^* within a continuous phase of liquid

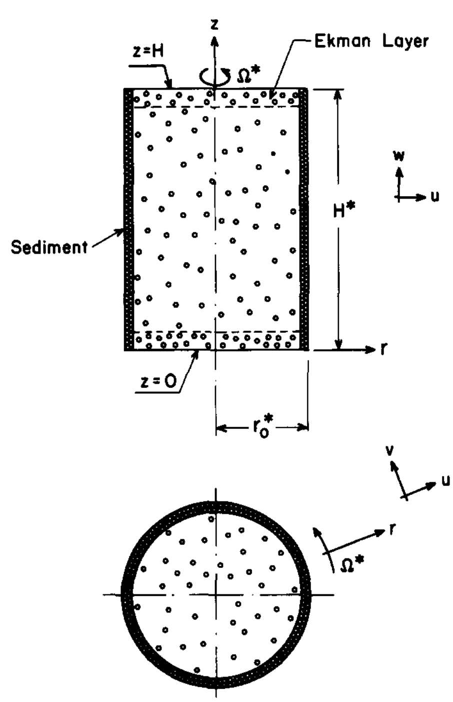

Figure 1. Description of the system.

of density ρ_C^* and viscosity μ_C^*. Initially, both phases are in solid rotation with the container whose angular velocity is Ω^*, and the gravity acceleration is negligible compared to $\Omega^{*2}r_0^*$.

The basic parameters of this problem are: the reduced density difference, $\epsilon = (\rho_D^* - \rho_C^*)/\rho_C^*$; the (modified) Taylor number of the dispersed particle, $\beta = \frac{2}{9}a^{*2}\Omega^*\rho_C^*/\mu_C^*$; the Ekman number, $E = \mu_C^*/\rho_C^*\Omega^* r_0^{*2}$; the aspect ratio (dimensionles height), $H = H^*/r_0^*$; and the initial disperse volume fraction, $\alpha(0)$.

Attention is focused on the range $E \ll 1$, $\beta \ll 1$, $|\epsilon| \sim 1$ and $H \gg E^{1/2}$. The physical implications of these restrictions will be clarified below.

The density difference between the phases (for definiteness, $\rho_D^* > \rho_C^*$) gives rise to a transient motion in which the heavier component is expelled towards the periphery. The objective is to investigate the velocity field and the volume fraction α of particles during this separation.

Important aspects of this process under closely related circumstances have been discussed by Greenspan (1983) for an *infinitely long* cylinder (endcaps neglected). An exact similarity solution of the two-fluid averaged equations of motion has been obtained whose main features are as follows. In the separating mixture bulk α is a function of t alone and the radial and azimuthal velocities vary linearly with the radius, r. The particles are squeezed from the mixture (there is no disengagement between phases in the central region when the inner radius is zero) and deposited on a thickening sediment layer on the outer wall of the container, where $\alpha = \alpha_M =$ const. This radial separation induces an azimuthal retrograde motion of magnitude $\sim |\epsilon|\alpha(0)\Omega^* r_0^*$ relative to a system rotating with Ω^*, see also Ungarish & Greenspan (1984). Since the radial velocities are $\sim |\epsilon|\beta\Omega^* r_0^*$, the significant deviation from solid rotation is represented by the Rossby number $\mathrm{Ro} = |\epsilon| \max(\alpha(0), \beta)$. When $|\epsilon| \ll 1$, Greenspan's (1983) solution provides simple analytical expressions for the radial and azimuthal components of the relative velocity between the phases.

The serious disadvantage of Greenspan's solution is the obvious omission of the shear layers on the endcaps of the container. This important effect has been investigated by Ungarish (1986) upon employing the *mixture* (or *diffusion*) formulation. The solution concerns the linear flow dominated by Coriolis terms which corresponds to the limit $\mathrm{Ro} \to 0$, i.e. $|\epsilon| \to 0$ for non-dilute suspensions. In addition, E and β are kept small to facilitate boundary layer analysis. The study has indicated that the viscous regions adjacent to the endplates have typical Ekman structure. They may considerably affect the azimuthal velocity of the inviscid core when the parameter $\lambda = E^{1/2}/|\epsilon|\beta H$ is not small. For instance, for $\lambda = 1$ the maximal azimuthal lag velocity is only about 35% of and decays faster than the value predicted by Greenspan (1983). An appropriate estimate yields

$$\mathrm{Ro} = |\epsilon| \max\left[\alpha(0)\left(\frac{1}{\lambda}\right)^{\lambda/(\lambda-1)}, \beta\right]$$

(cf. appendix A).†

However, in many cases of interest, ϵ is not infinitesimally small. In these circumstances the Ekman layers are expected to affect other flow variables in addition to the azimuthal velocity. In particular, it is worthy to question what are the consequences of these shear layers on the behavior of α and of the relative velocities, because these are the dominant quantities in numerous applications. The main objective of the present study is to provide quantitative and qualitative answers to this issue. Moreover, this investigation endeavors to clarify the compatibility between the *two-fluid* formulation and the *mixture* model (Ishii 1975) when applied to the present complicated flow field.

To this end, a solution of the *two-fluid* equations in the finite cylinder is sought. The Ekman layer effect is incorporated via an integral correlation, expected to hold for small (but finite) E and β. This procedure obviates both the *infinitely long* assumption of Greenspan (1983) and the *linear flow* approximation of Ungarish (1986).

2. FORMULATION

The time-dependent motion of the two incompressible, immiscible components in a system rotating with constant Ω^* is represented by the averaged variables of the continuous and dispersed

†Ungarish (1986) also considers a spatial initial variation of α, which is irrelevant to the present case.

phases, subscripted by C and D, respectively. The equations of motion in terms of the phase occupancy α_f, velocity $\mathbf{q}_f^*$, pressure p_f^* and stresses $\tilde{\tau}_f^*$ and $\tilde{\tau}_f^{*\mathrm{I}}$ (Ishii 1975) are, upon postulating a linear drag interaction:

$$\frac{\partial \alpha_f}{\partial t^*} + \nabla \cdot \alpha_f \mathbf{q}_f^* = 0, \qquad [1]$$

$$\alpha_f + \alpha_{f'} = 1 \qquad [2]$$

and

$$\rho_f^* \left[\frac{\partial \mathbf{q}_f^*}{\partial t^*} + \mathbf{q}_f^* \cdot \nabla \mathbf{q}_f^* + 2\mathbf{\Omega}^* \times \mathbf{q}_f^* \right] = -\nabla P_f^* - \frac{\alpha_\mathrm{D}}{\alpha_f} D^*(\mathbf{q}_f^* - \mathbf{q}_{f'}^*)$$
$$- (\rho_f^* - \rho_\mathrm{C}^*)\mathbf{\Omega}^* \times (\mathbf{\Omega}^* \times \mathbf{r}^*) + \nabla \cdot \tilde{\tau}_f^* + \frac{1}{\alpha_f} \tilde{\tau}_f^{*\mathrm{I}} \cdot \nabla \alpha_f, \qquad [3]$$

where the subscript f denotes the phases D or C, f' stands for the complementary component and P_f^* is the reduced pressure,

$$P_f^* = p_f^* - \tfrac{1}{2}\rho_\mathrm{C}^* |\mathbf{\Omega}^* \times \mathbf{r}^*|^2. \qquad [4]$$

To close the system it is further assumed that

$$P_\mathrm{D}^* - P_\mathrm{C}^* = \text{const}, \qquad [5]$$

the stress tensors are Newtonian and the drag term coefficient is Stokesian, i.e.

$$D^* = \mu_\mathrm{eff}^* \frac{9}{2} \frac{1}{a^{*2}}. \qquad [6]$$

The effective viscosity of the mixture μ_eff^* can be correlated, according to Ishii & Zuber (1979), as follows:

$$\frac{\mu_\mathrm{eff}^*}{\mu_\mathrm{C}^*} \triangleq \mu(\alpha) \approx (1 - \alpha)^{-2.5}. \qquad [7]$$

Here and thereafter, the notation $\alpha \equiv \alpha_\mathrm{D}$ is used. For later implementation, the mixture and relative velocities are also introduced,

$$\mathbf{q}_\mathrm{m} = \frac{[(1+\epsilon)\alpha\, \mathbf{q}_\mathrm{D} + (1-\alpha)\mathbf{q}_\mathrm{C}]}{(1+\epsilon\alpha)}, \quad \mathbf{q}_\mathrm{R} = \mathbf{q}_\mathrm{D} - \mathbf{q}_\mathrm{C}. \qquad [8]$$

The initial conditions are solid body rotation, $\mathbf{q}_f(\mathbf{r}, t=0) = 0$, with given homogeneous void fraction, $\alpha(r, t=0) = \alpha(0)$, through the container.

To be more specific, the cylindrical coordinate system (r, θ, z) rotating around z is employed and the variables are reduced to dimensionless form upon introducing the following scaling:

$$\mathbf{r} = \frac{\mathbf{r}^*}{r_0^*}, \quad t = t^* |\epsilon| \beta \Omega^*, \quad \mathbf{q} = \frac{\mathbf{q}}{(|\epsilon|\beta\Omega^* r_0^*)}, \quad \rho_f = \frac{\rho_f^*}{\rho_\mathrm{C}^*} \quad \text{and} \quad P_f = \frac{P_f^*}{(\frac{1}{2}\rho_\mathrm{C}^* |\epsilon| \Omega^{*2} r_0^{*2})}.$$

The solution to [1]–[7] is sought as a superposition of an (almost) inviscid core and thin viscous layers on the endplates. *For the main core*, the following expansion is introduced:

$$\mathbf{q}(\mathbf{r}, t) = \left[r U_f^{(0)}(t)\hat{r} + r V_f^{(0)}(t)\hat{\theta} + \mathrm{E}^{1/2} W_f^{(0)}(t)\left(\frac{2z}{H} - 1\right)\hat{z} \right]$$
$$+ \mathrm{E}^{1/2}[u_f^{(1)}\hat{r} + v_f^{(1)}\hat{\theta} + \mathrm{E}^{1/2} w_f^{(1)}\hat{z}] + \cdots, \qquad [9]$$

$$\alpha = \alpha^{(0)}(t) + \mathrm{E}^{1/2}\alpha^{(1)} + \cdots \qquad [10]$$

and

$$P_f(\mathbf{r}, t) = r^2 P^{(0)}(t) + \mathrm{E}^{1/2} P^{(1)} + \cdots + \text{const}. \qquad [11]$$

Only the leading terms are subsequently retained and the superscript (0) is therefore dropped.

Substituting [9]–[11] into [1]–[7] yields: *for the core region* the equations of continuity,

$$\alpha' + 2\alpha\left(U_D + \frac{E^{1/2}}{H} W_D\right) = 0 \qquad [12]$$

and

$$-\alpha' + 2(1-\alpha)\left(U_C + \frac{E^{1/2}}{H} W_C\right) = 0; \qquad [13]$$

the radial and azimuthal momentum equations for the dispersed phase,

$$\beta(1+\epsilon)[\beta|\epsilon|(U'_D + U_D^2 - V_D^2) - 2V_D] = -P + \mu(\alpha)(U_C - U_D) + \frac{\epsilon}{|\epsilon|} \qquad [14]$$

and

$$\beta(1+\epsilon)[\beta|\epsilon|(V'_D + 2U_D V_D) + 2U_D] = \mu(\alpha)(V_C - V_D); \qquad [15]$$

and the radial and azimuthal equations for the continuous phase,

$$\beta[\beta|\epsilon|(U'_C + U_C^2 - V_C^2) - 2V_C] = -P - \frac{\alpha}{1-\alpha}\mu(\alpha)(U_C - U_D) \qquad [16]$$

and

$$\beta[\beta|\epsilon|(V'_C + 2U_C V_C) + 2U_C] = -\frac{\alpha}{1-\alpha}\mu(\alpha)(V_C - V_D); \qquad [17]$$

here the prime denotes the differentiation in time.

The leading balance for axial momentum yields identical results for both phases, namely,

$$W_D = W_C (= W). \qquad [18]$$

An additional equation is therefore necessary for defining the axial velocity component W. It is postulated that the axial velocity is induced by the viscous Ekman layers on the top and bottom plates. Guided by analogy with homogeneous fluids (Wedemeyer 1964), which is apparently supported by Ungarish (1986) for $|\epsilon| \to 0$ and by numerical solutions in the more general case (Ungarish 1988), it seems reasonable to employ the approximation

$$W(t) = -\sqrt{\mu(\alpha)}\, V_m(t), \qquad [19]$$

where

$$V_m = \frac{[(1+\epsilon)\alpha V_D + (1-\alpha)V_C]}{(1+\epsilon\alpha)} \qquad [20]$$

is the angular velocity of the mixture in the core. Actually, the Ekman layers are expected to develop in about one revolution of the system which corresponds to the short time interval $t \sim |\epsilon|\beta$ in the present scaling and are subsequently quasi-steady. Approximation [19] implies the condition that the typical boundary layer thickness, $\sqrt{\mu_C^*/\rho_C^*\Omega^*}$, is able to encompass many particle radii, a^*. This is equivalent to the restriction $\beta \ll 1$.

The formulation of the internal core flow, to leading order in $E^{1/2}$, is now complete. This is an initial-value problem, starting with zero velocity components and the prescribed $\alpha(0)$.

The task now is to calculate the solution for the time-dependent variables α, U_D, U_C, V_D and V_C from the preceding equations. Equations [12], [15] and [17], after rearrangement, define α', V'_D and V'_C in standard form. The radial momentum equations require some manipulation. First, using [12] and [13] one gets

$$U_C = \frac{-1}{1-\alpha}\left(\alpha U_D + \frac{E^{1/2}}{H} W\right), \qquad [21]$$

whose time derivative in view of [19] is

$$U_C' = \frac{-1}{1-\alpha}\left\{\alpha'(U_D - U_C) + \alpha U_D' - \frac{E^{1/2}}{H}\left[\frac{1}{2\sqrt{\mu(\alpha)}}\frac{d\mu}{d\alpha}\alpha' V_m + \sqrt{\mu(\alpha)}\, V_m'\right]\right\}, \qquad [22]$$

and V_m' is obtained from [20] as an explicit function of α, V_D and V_C and their first time derivative. Next, eliminating P from [14] and [16] and further replacing U_C' by [22] yields an equation for U_D' in terms of α, U_D, U_C, V_D, V_C, α', V_D' and V_C'. Since the latter three derivatives have already been expressed in standard form, the subsequent calculations are routine numerical work (see appendix B).

It is worthwhile to remark that eliminating the axial component from [12] and [13] on account of [18] gives

$$\alpha' + 2\alpha(1-\alpha)(U_D - U_C) = 0, \qquad [23]$$

which emphasizes that the variation of α in the core is dominated by the relative motion in the radial direction. This motion, however, is indirectly affected by the axial Ekman layer suction, as shown below.

3. RESULTS

The foregoing system for the flow variables in the interior mixture core has been integrated for various combinations of ϵ, β and λ (recall, $E^{1/2}/H = \lambda|\epsilon|\beta$); for definiteness, $\alpha(0) = 0.2$ was used in these computations. The corresponding predictions of Greenspan (1983) and Ungarish (1986) serve for comparison. Greenspan's solution is straightforwardly reproduced, letting $\lambda = 0$. However, Ungarish's results represent the singular limit $\epsilon \to 0$ of the present model and required, consequently, the employment of the particular formulation, appendix A.

Figure 2 emphasizes the important effect played by the Ekman layers on the angular velocity of the mixture, for a system with $\beta = 0.1$ and various ϵ and λ. The predictions of Ungarish in this respect are clearly confirmed, with quite small modifications induced by the non-linear convective contribution for non-zero ϵ.

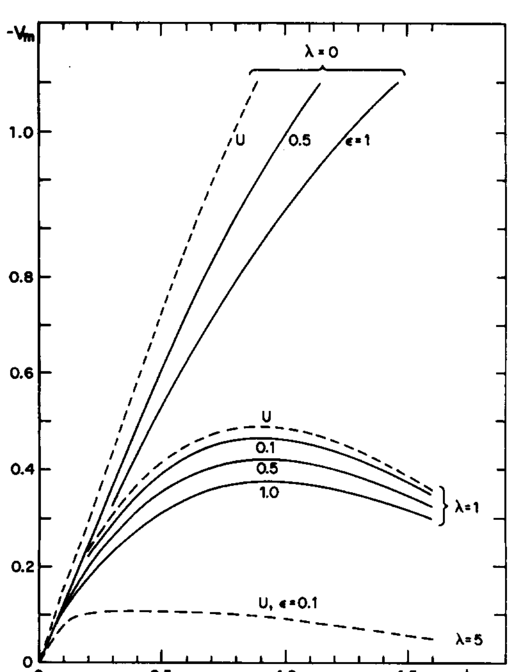

Figure 2. Angular velocity of the mixture vs time for various ϵ and λ ($\lambda = 0$ corresponds to Greenspan's (1983) solution and Ungarish's (1986) results, marked U, represent the limit $\epsilon \to 0$). $\beta = 0.1$, $\alpha(0) = 0.2$.

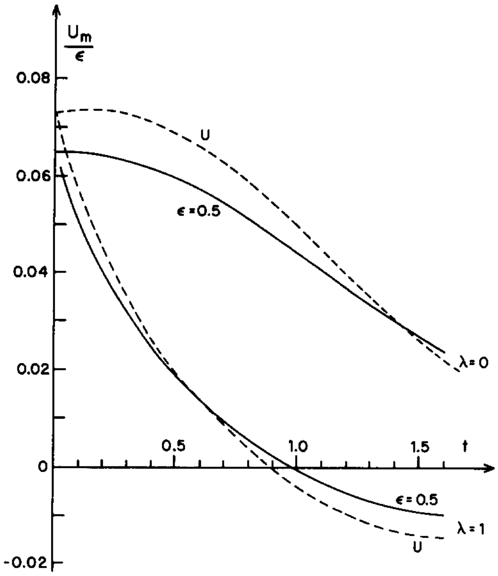

Figure 3. Reduced radial velocity of the mixture vs time. $\beta = 0.1$, $\alpha(0) = 0.2$.

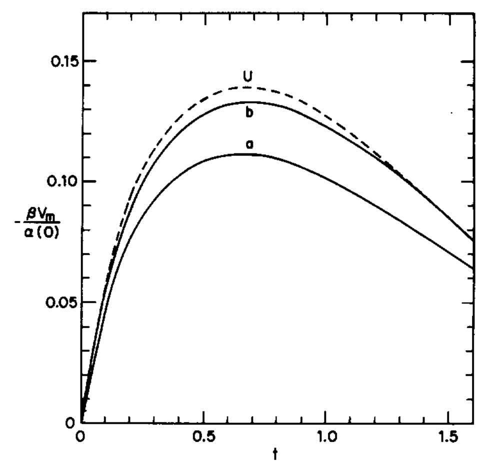

Figure 4. Reduced angular velocity vs time for different systems with $\lambda = 2$ and $\alpha(0) = 0.2$: (a) $\epsilon = 0.3$, $\beta = 0.2$; (b) $\epsilon = 0.2$, $\beta = 0.05$.

The typical behavior of the radial velocity of the mixture is displayed in figure 3.† It is observed that U_m is indeed proportional to ϵ and that, for non-small λ, the Ekman layers eventually give rise to a reverse mass flow in the core.

Figure 4 intends to further test the relevance of λ. This parameter appears explicitly in the analysis of Ungarish (1986), but is apparently obscured by the more complete present formulation. Reduced angular velocities vs time for several different combinations of ϵ and β, but fixed $\lambda = 2$ (i.e. properly chosen values of $(E^{1/2}/H) = \lambda |\epsilon| \beta)$ are drawn. The different graphs display a very similar behavior and tend to collapse into Ungarish's (1986) solution when $|\epsilon|$ diminishes. Similar features have been observed for other values of λ, not shown here.

So far, the main relevant predictions of Ungarish (1986) have been confirmed. The next important topic is establishing the influence of non-zero λ on the behavior of α and the components of the relative velocity $\mathbf{q}_R = \mathbf{q}_D - \mathbf{q}_C$. The typical system $\beta = 0.1$, $\epsilon = 0.5$, $\lambda = 1$ is considered.

Figures 5 and 6 indicate that, in comparison to Greenspan's (1983) solution, the present values of α decay faster and the relative velocities in both the radial and azimuthal directions eventually become larger.‡ This trend is monotonical in time. However, as predicted by Greenspan, U_R and $(-V_R/2\beta)$ are of comparable magnitude. For a better understanding of these features it is worthwhile to concentrate on the discrepancies of the relative velocities, in comparison with the case $\lambda = 0$, see figure 7. It is argued that these discrepancies are the induced result of the different angular velocities, see figure 2, for the following reasons. First, recall that the absolute effective angular velocity of the mixture is

$$\Omega^*_{\text{eff}} = \Omega^*(1 + |\epsilon| \beta V_m). \quad [24]$$

Next, observe that physical considerations imply that the relative velocities of the small dispersed particles are dominated by local balances. The drag components counteract the radial buoyancy and the azimuthal Coriolis terms; therefore

$$U_R \propto \Omega^{*2}_{\text{eff}} \quad [25]$$

and

$$V_R \propto -\Omega^*_{\text{eff}} U_R. \quad [26]$$

†The initial behavior of this variable is beyond the resolution of this graph, see appendix C.
‡The footnote above applies to V_R and U_R.

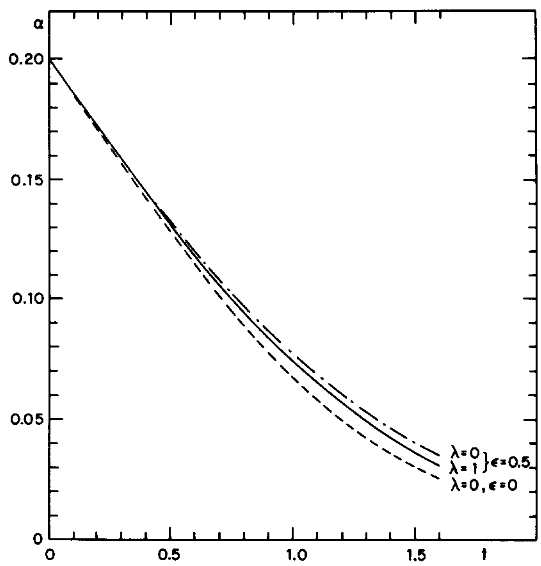

Figure 5. α vs time.

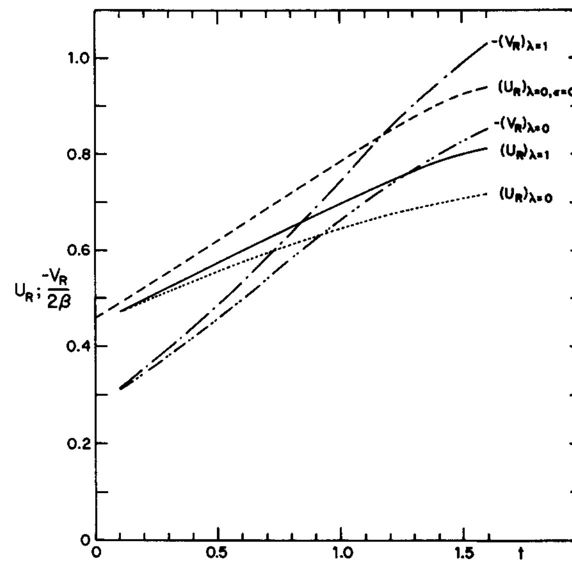

Figure 6. Radial and azimuthal components of the relative velocity, $\mathbf{q}_R = \mathbf{q}_D - \mathbf{q}_C$, vs time for $\lambda = 1$ and $\lambda = 0$. $\beta = 0.1$, $\epsilon = 0.5$ (except as noted).

Combining [24]–[26] yields

$$\frac{[U_R]_\lambda}{[U_R]_{\lambda=0}} - 1 \approx 2|\epsilon|\beta([V_m]_\lambda - [V_m]_{\lambda=0}) \qquad [27]$$

and

$$\frac{[V_R]_\lambda}{[V_R]_{\lambda=0}} - 1 \approx 3|\epsilon|\beta([V_m]_\lambda - [V_m]_{\lambda=0}). \qquad [28]$$

These formulas reproduce both qualitatively and quantitatively the calculated discrepancies, see figure 7, which indicates that [24]–[26] indeed account for the pertinent leading mechanism.

Since $0 \leqslant [V_m]_\lambda - [V_m]_{\lambda=0} \leqslant \alpha(0)/\beta$ it is concluded that the relative discrepancies of U_R and V_R from Greenspan's solution are non-negative and (approximately) bounded by $2|\epsilon|\alpha(0)$ and $3|\epsilon|\alpha(0)$, respectively; the bound is approached for large λ.

In view of the foregoing results and [23] it becomes evident that α decays faster for larger λ, as indeed observed in figure 5. Moreover, it is realized that, actually, [27] and [28] give underestimates, because U_R and V_R are also proportional to a hindering function of the form $\sim (1-\alpha)^{4.5}$. This coefficient, at a particular time instant, increases with λ due to the faster decay of α.

The discrepancy of U_D with $[U_D]_{\lambda=0}$ is more complicated, see figure 8. A large λ causes an initial reduction of the dispersed-phase radial velocity, but at a later stage this velocity difference changes sign. To explain this feature consider the kinematic relationship

$$U_D = J_r + (1-\alpha)U_R, \qquad [29]$$

where $J_r = \alpha U_D + (1-\alpha)U_C$ is the reduced volume flux in the radial direction. The Ekman layers drive fluid to the periphery and volume conservation therefore requires a negative J_r in the core, but for Greenspan's solution (i.e. $\lambda = 0$), this flux is identically zero. Therefore,

$$[U_D]_\lambda - [U_D]_{\lambda=0} = [J_r]_\lambda + (1-\alpha)([U_R]_\lambda - [U_R]_{\lambda=0}). \qquad [30]$$

The first term on the r.h.s. is negative but eventually decays while the positive latter one monotonically grows with t, resulting in the typical behavior observed in figure 8.

The foregoing analysis, and especially [27], leads to the conclusion that, for given ϵ and β, the rate of separation improves with increasing λ (for instance, by employing a shorter container, all other variables unchanged). The relative enhancement potential is $\sim 2|\epsilon|\alpha(0)$ as compared to Greenspan's solution (1983).

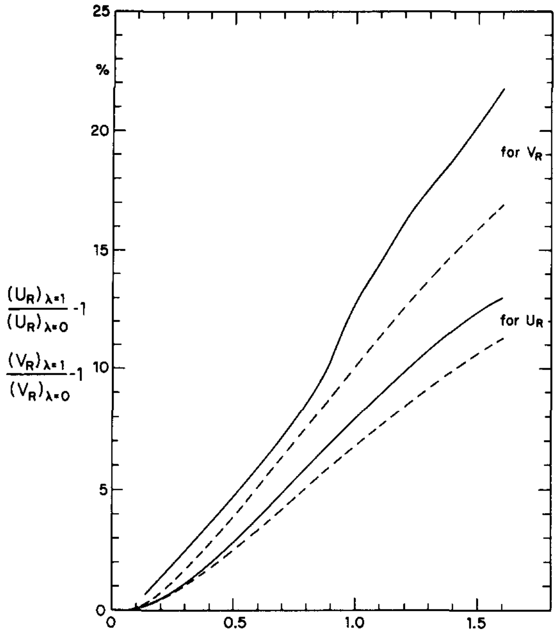

Figure 7. Percentage discrepancies of the relative velocity components with Greenspan's solution vs time with $\beta = 0.1$, $\epsilon = 0.5$ and $\lambda = 1$: ——, direct computation; ------, [27] and [28].

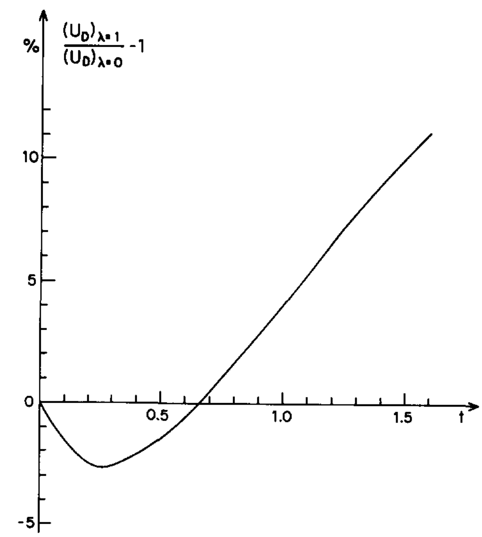

Figure 8. Percentage discrepancy of U_D with Greenspan's solution vs time with $\beta = 0.1$, $\epsilon = 0.5$ and $\lambda = 1$.

4. CONCLUDING REMARKS

A solution method for the *two-fluid* equations of motion in the almost inviscid separating mixture core in a *finite* rotating cylinder has been developed. The results further elucidate the effects of the Ekman layers on the flow field.

The asymptotic Ro → 0 (i.e. $|\epsilon| \to 0$) indications of the finite cylinder treatment (Ungarish 1986) concerning the dominance of parameter λ on the azimuthal motion have been verified and extended to non-linear flows. Thus, in general, the azimuthal retrograde $O(|\epsilon|\alpha(0)\Omega^*)$ angular velocity predicted by the *infinitely long* cylinder approach (Greenspan 1983) is substantially reduced when $\lambda \sim 1$ and diminishes when λ is large. This gives rise to a corresponding relative increase of the effective absolute angular velocity which induces larger relative velocity components and a faster local separation rate ($d\alpha/dt$) than in the "infinite" cylinder. It is emphasized, however, that for non-small λ the relative discrepancies with Greenspan's solution are O(1) in azimuthal velocity and $O(|\epsilon|\alpha(0))$ for the volume fraction and relative velocity.

The present solution recovers exactly the results of Greenspan (1983) when $\lambda = 0$ and those of Ungarish (1986) when $\epsilon \to 0$ with, essentially, comparable computational resources. Nevertheless, the relative merits of the previous models should not be ignored. Thus, it is briefly recalled that Greenspan's solution is "exact" (for $\lambda = 0$), provides simple closed-form relationships between the positions of the kinematic shocks and $\alpha(t)$ and for the relative velocity components when $\epsilon = 0$ and reveals the initial behavior on the "relaxation" scale. However, Greenspan's solution is restricted to $\alpha(\mathbf{r}, 0) = \text{const}$ and to $\lambda \ll 1$. Ungarish's approximation is facile for analytic manipulations, highlights parametric dependencies and solves the details of the Ekman layer flow. Its limitations are the Ro ≪ 1 (i.e. $|\epsilon| \ll 1$) assumption and the employment of a closure formula for the relative velocity. Finally, it is noted that the present approach is superior because it covers a wider range of parameters than either Greenspan's or Ungarish's. The main disadvantages are: this solution concerns only the leading term in an $E^{1/2}$ powers expansion, it relies on an integral Ekman layer suction premise and is restricted to $\alpha(\mathbf{r}, 0) = \text{const}$.

The good compatibility between Ungarish's *mixture* (*diffusion*) results and the present *two-fluid* computations is very satisfactory. This enhances the confidence in the former formulation as a means for investigating more complicated configurations, where the latter modeling may become quite cumbersome.

This study concerns the (almost inviscid) mixture core only. A sediment layer and a pure fluid region (if the internal radius of the container is not zero) are expected to develop, to leading order in $E^{1/2}$, in close resemblance to Greenspan's solution. The additional details, which probably require the solution of $E^{1/4}$ and $E^{1/3}$ shear layers, are not pursued here.

The linear Stokesian relation between drag and relative velocity can be justified when $\beta \ll 1$. This restriction also assures two pertinent conditions concerning the Ekman layers: their being quasi-steady on the separation time scale and much wider than the particle diameter.

The Ekman layer suction correlation [19], which is the most critical step of the present theory, deserves a great deal of both attention and suspicion. Actually, very little is known about these layers in the two-phase non-linear regime [the finite difference computation (Ungarish 1988) and the related von Karman solution (Ungarish & Greenspan 1983), indicate similarities to the single-phase flow but also reveal some dilemmas of formulation and interpretation]. In this respect, the present results relate the more observable core flow to the smaller and less observable circulation in the thin shear regions. This will, hopefully, facilitate experimental verifications to throw more light on this important and difficult topic.

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APPENDIX A

Equations and Estimations of Ungarish (1986) for the Core Region

Postulate:

$$\mathbf{q}_R = s\frac{1-\alpha}{\mu(\alpha)} r\,\hat{r},$$

where

$$s = \frac{\epsilon}{|\epsilon|}.$$

Equation of motion:

$$\alpha' = \frac{-s\alpha(1-\alpha)^2}{\mu(\alpha)} \qquad [A.1]$$

and

$$(\beta V_m)' = -2\lambda\sqrt{\mu(\alpha)}(\beta V_m) + s\alpha'. \qquad [A.2]$$

In addition,

$$\left(\frac{U_m}{|\epsilon|}\right) = \frac{\alpha(1-\alpha)^2}{\mu(\alpha)} + \lambda\sqrt{\mu(\alpha)}(\beta V_m) \qquad \text{[A.3]}$$

and

$$\left(\frac{J_r}{|\epsilon|}\right) = \lambda\sqrt{\mu(\alpha)}(\beta V_m). \qquad \text{[A.4]}$$

Thus, for given λ and $\alpha(0)$, one obtains generalized solutions for (βV_m), $(U_m/|\epsilon|)$ and $(J_r/|\epsilon|)$ vs t.

In the dilute limit estimate, the extremum

$$(\beta V_m) = -\alpha(0)\left(\frac{1}{\lambda}\right)^{\lambda/(\lambda-1)}$$

$(= -\alpha(0)/\mathrm{e}$ when $\lambda \to 1)$ is reached at

$$t = \frac{\ln\lambda}{2(\lambda-1)} \quad (=\tfrac{1}{2} \text{ when } \lambda \to 1).$$

APPENDIX B

The standard initial-value system for α, U_D, V_D, U_C and V_C can be expressed as follows:

$$\alpha' = -2\alpha(1-\alpha)(U_D - U_C); \qquad \text{[B.1]}$$

$$V'_D = \frac{1}{|\epsilon|\beta}\left[\frac{\mu(\alpha)}{\beta(1+\epsilon)}(V_C - V_D) - 2U_D(1+|\epsilon|\beta V_D)\right]; \qquad \text{[B.2]}$$

$$V'_C = \frac{-1}{|\epsilon|\beta}\left[\frac{\alpha\mu(\alpha)}{\beta(1-\alpha)}(V_C - V_D) + 2U_C(1+|\epsilon|\beta V_C)\right]; \qquad \text{[B.3]}$$

$$V_m = \frac{[\alpha(1+\epsilon)V_D + (1-\alpha)V_C]}{(1+\epsilon\alpha)}; \qquad \text{[B.4]}$$

$$V_{mt} = \frac{\{(1+\epsilon)\alpha V'_D + (1-\alpha)V'_C + \alpha'[(1+\epsilon)V_D - V_C - \epsilon V_m]\}}{(1+\epsilon\alpha)}; \qquad \text{[B.5]}$$

$$W_t = -\frac{\mathrm{E}^{1/2}}{H}\left[\frac{\mathrm{d}(\sqrt{\mu})}{\mathrm{d}\alpha}\alpha' V_m + \sqrt{\mu(\alpha)}V_{mt}\right]; \qquad \text{[B.6]}$$

$$X = (1-\alpha)\left\{\frac{-\alpha'(U_D - U_C)}{(1-\alpha)} + U_C^2 - V_C^2 - (1+\epsilon)(U_D^2 - V_D^2) + \frac{2}{\beta|\epsilon|}[(1+\epsilon)V_D - V_C] + \frac{1}{\beta^2|\epsilon|}\left[\frac{\epsilon}{|\epsilon|} + \frac{\mu(\alpha)}{1-\alpha}(U_C - U_D)\right]\right\}; \qquad \text{[B.7]}$$

$$U'_D = \frac{1}{1+\epsilon(1-\alpha)}(X - W_t); \qquad \text{[B.8]}$$

and

$$U_C = \frac{1}{1-\alpha}\left[-\alpha U_D + \frac{\mathrm{E}^{1/2}}{H}\sqrt{\mu(\alpha)}V_m\right]. \qquad \text{[B.9]}$$

Here the prime denotes differentiation with respect to time. Note that [B.4]–[B.7] and [B.9] represent algebraic relationships needed for the integration of the primed variables.

APPENDIX C

Relaxation Time Boundary Layer

A simple inspection of the momentum equations [14]–[17] indicates that an initial singular behavior occurs when $|\epsilon|\beta^2 \to 0$. During this short relaxation time interval the contribution of the Ekman layers is negligible and α is almost constant, therefore good analytic approximations can be obtained, e.g.

$$U_{\mathrm{D}} = \frac{\epsilon}{|\epsilon|} K_1 \left[1 - \exp\left(-\frac{K_2}{|\epsilon|\beta^2} t\right)\right], \qquad U_{\mathrm{m}} = \frac{\epsilon\alpha(0)\, U_{\mathrm{D}}}{1 + \epsilon\alpha(0)}, \qquad U_{\mathrm{R}} = \frac{U_{\mathrm{D}}}{1 - \alpha(0)}$$

and

$$V_{\mathrm{m}} = -\frac{2}{\beta}\alpha(0)\frac{K_1}{1 + \epsilon\alpha(0)}\left\{t + \frac{|\epsilon|\beta^2}{K_2}\left[\exp\left(-\frac{K_2}{|\epsilon|\beta^2} t\right) - 1\right]\right\},$$

where

$$K_1 = \frac{[1 - \alpha(0)]^2}{\mu[\alpha(0)]} \quad \text{and} \quad K_2 = \frac{1}{K_1}\frac{1 - \alpha(0)}{1 + \epsilon[1 - \alpha(0)]}.$$